UWI Math Fair 2019

Sample Math Talent Exam - Solutions

1. How many integers from 1 to 2019 (inclusive) are multiples of 3 but not 4?

Answer:

The multiples of 3 are 3, 6, 9, ..., 2019, so the number of multiples of 3 is $\frac{2019}{3} = 673$.

We need to exclude from this number, the number of integers that are multiples of both 3 and 4, i.e. multiples of 12.

The multiples of 12 are 12, 24, 36, ..., 2016,The number of multiples of 12 is $\frac{2016}{12} = 168$.

Therefore, the answer is $673 - 168 = \boxed{505}$.

Marking Scheme:

- Calculating that there are 673 multiples of 3. **[1]**
- Stating that we must (or attempting to) exclude multiples of 12. **[1]**
- Calculating that there are 168 multiples of 12. **[1]**
- Subtracting both numbers. **[1]**
- Stating that the answer is 505. **[1]**

2. In a certain group of students, 60% of the boys play cricket and 30% of the girls play cricket. The number of girls in the group is twice the number of boys. What percentage of the students in the group do not play cricket?

Answer:

Let x be the total number of students. The ratio of girls to boys is 2 to 1, so $\frac{2}{3}x$ are girls and $\frac{1}{3}x$ are boys.

The number of boys who play cricket is $60\% \times \frac{1}{3}x = \frac{1}{5}x$.

The number of girls who play cricket is $30\% \times \frac{2}{3}x = \frac{1}{5}x$.

The total number of students who play cricket is $\frac{1}{5}x + \frac{1}{5}x = \frac{2}{5}x$.

Therefore, the number who do not play cricket is $x - \frac{2}{5}x = \frac{3}{5}x$.

The percentage who do not play cricket is $\frac{\frac{3}{5}x}{x} \times 100\% = \boxed{60\%}$.

Marking Scheme:

- Stating that of the students, $\frac{2}{3}$ are girls and $\frac{1}{3}$ are boys. **[1]**
- Obtaining that $\frac{1}{5}$ are girls and play cricket. **[1]**
- Obtaining that $\frac{1}{5}$ are boys and play cricket. **[1]**
- Obtaining that $\frac{3}{5}$ do not play cricket. **[1]**
- Stating that the answer is 60%. **[1]**

3. Solve for A, B and C given that:

$$\begin{aligned} 2A + B - 3C &= 9, \qquad (1) \\ B \times C &= 10A, \qquad (2) \\ 2A &= B. \qquad (3) \end{aligned}$$

Answer:

Substituting for B from (3) into (1) and (2), respectively, gives

$$\begin{aligned} 4A - 3C &= 9 \qquad (4) \\ 2A \times C &= 10A \qquad (5) \end{aligned}$$

Equation (5) can be rewritten as

$$2A\,(C-5) = 0,$$

so either $A = 0$ or $C = 5$.

If $A = 0$, then from (3), $B = 0$ and from (4), $C = -3$.

If $C = 5$, then from (4), $4A - 15 = 9$, so $A = \frac{9+15}{4} = 6$. This gives $B = 2 \times 6 = 12$.

Therefore, the solutions are: $\boxed{A = 0,\ B = 0,\ C = -3}$ and $\boxed{A = 6,\ B = 12,\ C = 5}$.

Marking Scheme:

- Correctly eliminating one variable. **[1]**
- Reducing the problem to considering two cases. **[1]**
- Correctly addressing the case $A = 0$ or $B = 0$. **[1]**
- Correctly addressing the other case. **[1]**
- Stating both correct answers. **[1]**

4. Let ABC be an isosceles triangle with $AB = AC$. Consider the circle passing through the vertices of triangle ABC. Suppose that the tangent to this circle at point B is perpendicular to the line AC. Find $\angle ABC$.

Answer:

Let $\angle ABC = \alpha$.

Then $\angle ACB = \alpha$ since $AB = AC$, so $\angle BAC = 180° - 2\alpha$.

Let X be the point of intersection of the tangent and line AC.

By the Alternate Segment Theorem,

$$\angle XBA = \angle ACB = \alpha. \tag{6}$$

Also, $\angle BAX = 180° - \angle BAC = 2\alpha$.

Since $\angle BXA = 90°$,

$$\angle XBA = 90° - 2\alpha. \tag{7}$$

Equating (6) and (7) gives $\alpha = 90° - 2\alpha$, so $\alpha = \boxed{30°}$.

Marking Scheme:

- Obtaining $\angle BAC = 180° - 2\angle ABC$ or an equivalent statement **[1]**
- Obtaining $\angle XBA = \angle ACB$. **[1]**
- Obtaining $\angle XBA = 90° - 2\angle ABC$ or an equivalent statement. **[1]**
- Equating (6) and (7), or equivalent statements. **[1]**
- Stating that the answer is 30°. **[1]**

5. Water can be poured into a particular tank using a cold water tap and a hot water tap, and can be drained from the tank using a hole at the base of the tank. With the cold water tap open, and the hot water tap and the hole both closed, the tank will fill up in 14 minutes. With both taps closed, the full tank can be emptied in 21 minutes by opening the hole. If the hole is open, and the cold water tap and the hot water tap are opened simultaneously, then the tank will fill up in 12.6 minutes. Determine the number of minutes needed to fill up the tank with hot water, assuming at first the tank is empty and the hole and cold water tap are both closed.

Answer:

Let L units^3 be the capacity of the tank, and x be number of minutes required by the hot water tap to fill up the tank. Then the rate at which the hot water tap fills the tank is $\frac{L}{x}$ per minute.

The rate at which the cold water tap fills the tank is $\frac{L}{14}$ per minute.

The rate at which the hole "fills" the tank is $-\frac{L}{21}$ per minute.

With all three things open, the tank fills up in 12.6 minutes, so the sum of the above rates is $\frac{L}{12.6}$, i.e.

$$\frac{L}{x} + \frac{L}{14} - \frac{L}{21} = \frac{L}{12.6} \tag{8}$$

$$\implies \quad \frac{1}{x} + \frac{1}{14} - \frac{1}{21} = \frac{1}{12.6}$$

$$\implies \quad x = \frac{1}{\frac{1}{12.6} - \left(\frac{1}{14} - \frac{1}{21}\right)} = \frac{1}{\frac{1}{12.6} - \frac{1}{42}} = \frac{42}{\frac{1}{0.3} - 1} = \boxed{18}$$

Marking Scheme:

- Obtaining equation (8) or an equivalent equation. **[3]**
- Showing that the answer is 18 (with working). **[2]**

6. Evaluate the expression

$$1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \ldots + 2017 - 2018 + 2019.$$

Answer:

$$\begin{aligned}
&1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \ldots + 2017 - 2018 + 2019 \\
=&(1-2) + (3-4) + (5-6) + (7-8) + \ldots + (2017-2018) + 2019 \\
=&\underbrace{(-1) + (-1) + (-1) + (-1) + \ldots + (-1)}_{\frac{2017-1}{2}\text{ times}} + 2019 \\
=&-1009 + 2019 \\
=&\boxed{1010}
\end{aligned}$$

Marking Scheme:

- Considering a partition of the numbers that would correctly simplify the calculation. **[1]**
- Using the partition to get an answer. **[3]**
- Stating that the answer is 1010. **[1]**

7. Two players A and B participate in the following game:

   Initially there is a pile of 2018 stones. A plays first, choosing a positive divisor of 2018 and removing that number of stones from the pile. Then B picks a positive divisor of the number of remaining stones, and removes that number of stones from the pile, and so on. The player who removes the last stone loses.

   Prove that one of the players has a winning strategy and describe it.

Answer:

A has a winning strategy.

A wins if he removes one stone each time, leaving an odd number of stones for B's turn.

Each time B makes a move, he will have to choose an odd divisor since an odd number cannot have even divisors. Since the difference of two odd numbers is even, B will always have to leave an even number of stones for A's turn.

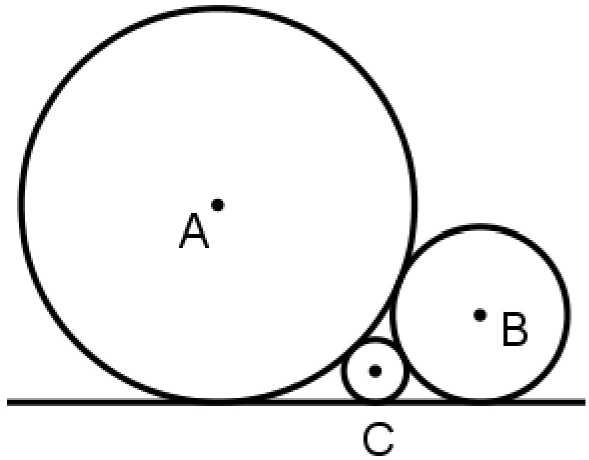
Since 1 is odd, B can never leave 1 stone for A.

The number of stones in the pile reduces each turn, so eventually A will leave 1 stone for B, so B will lose.

Marking Scheme:

- Stating that A has a winning strategy. **[1]**
- Stating that A should remove 1 stone each turn. **[1]**
- Stating that this will always leave B with an odd number of stones. **[1]**
- Stating that once A does this, B can never leave A with an odd number of stones. **[1]**
- Concluding that B will have to remove the last stone. **[1]**

8. Three circles are tangent to each other and to a straight line, as shown below. The radii of the circles with centres A, B and C are a, b and c units respectively. Express c in terms of a and b.



Answer:

Note that $AB = a + b$, $AC = a + c$ and $BC = b + c$.

Shift the straight line c units up, so that it is parallel to the original line and passes through C. Let X and Y be the feet of the perpendiculars from A and B, respectively, to this new straight line.

Note that $XA = a - c$ units and $YB = b - c$ units.

Using Pythagoras' Theorem in triangles AXC and CYB, respectively, gives

$$XC = \sqrt{(a + c)^2 - (a - c)^2} = \sqrt{4ac}$$

and

$$CY = \sqrt{(b + c)^2 - (b - c)^2} = \sqrt{4bc}.$$

Now, shift the original straight line b units up, so that it is parallel to the original line and passes through B. Let Z be the foot of the perpendicular from A to this new straight line.

Note that $XZ = b - c$ and $AZ = a - b$.

Also,

$$ZB = XC + CY = \sqrt{4ac} + \sqrt{4bc}.$$

Using Pythagoras' Theorem in triangle AZB gives

$$(a + b)^2 - (a - b)^2 = \left(\sqrt{4ac} + \sqrt{4bc}\right)^2$$

$$\implies \quad 4ab = \left(2\sqrt{c}\left(\sqrt{a}+\sqrt{b}\right)\right)^2$$

$$\implies \quad 4ab = 4c\left(\sqrt{a}+\sqrt{b}\right)^2$$

$$\implies \quad \boxed{c = \frac{ab}{\left(\sqrt{a}+\sqrt{b}\right)^2}}$$

Marking Scheme:

- Obtaining $XC = \sqrt{4ac}$ and $CY = \sqrt{4bc}$. **[3]**
- Using these to obtain an answer. **[1]**
- Stating that the correct answer is $c = \frac{ab}{\left(\sqrt{a}+\sqrt{b}\right)^2}$. **[1]**